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ON ACTIONS OF ADJOINT TYPE ON COMPLEX STIEFEL MANIFOLDS

BY

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ABSTRACT. Let $G(m)$ denote SU(m) or Sp(m). It is shown that when $m \geq 5$ $G(m)$ cannot act smoothly on $W_{n,2}$, the complex Stiefel manifold of orthonormal 2-frames in $\mathbf{C}^n$, for n odd, with connected principal isotropy type equal to the class of maximal tori in $G(m)$. This demonstrates an important difference between $W_{n,2}$, n odd, and $S^{2n-3} \times S^{2n-1}$ in the behavior of differentiable transformation groups. Exactly the same holds for SO(m) or Spin(m) if it is further assumed that a maximal 2-torus of SO(m) has fixed points.[2]

Introduction. This is the second in a series of papers devoted to understanding properties of compact differentiable transformation groups on the Stiefel manifolds. In [**12**], the first paper in this series, we explained at some length the motivation behind such a project. Here we continue the investigations begun in [**12**]. While the results in this paper are independent of those in [**12**], we nevertheless assume some familiarity with the notation and basic definitions described in the introduction there.

To explain our results let us recall that $W_{n,2}$ $(= X)$, the complex Stiefel manifold of orthonormal 2-frames in $\mathbf{C}^n$, is of the same rational homotopy type (also integral cohomology type) as $S^{2n-3} \times S^{2n-1}$. However, they are not of the same homotopy type. When n is odd, this difference in homotopy type is detected by Sq^2.

Turning to transformation groups, let us recall that the principal isotropy type of the adjoint representation of a compact connected Lie group is the conjugacy class of the maximal tori. On $S^{2n-3} \times S^{2n-1}$ we may easily construct smooth actions of $G(m) = \mathrm{SU}(m)$, $\mathrm{Sp}(m)$, $\mathrm{SO}(m)$, or $\mathrm{Spin}(m)$ with principal isotropy type (T), where T denotes a maximal torus of $G(m)$: we just let $G(m)$ act trivially on S^{2n-3} and act via the restriction of the representation Ad $\oplus$ trivial on $\mathbf{R}^{2n}$ to the unit sphere S^{2n-1}. (Of course this necessitates $\dim G(m) \leq 2n$.)

The most natural way of defining a smooth action of $G(m)$ on X based on the adjoint representation of $G(m)$ is via the complexification of the representations

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[2]Any Spin(m) action of the type under consideration necessarily factors through SO(m), as shall be shown.

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$k\,\mathrm{Ad} \oplus$ trivial. In other words, let $N = \dim G(m)$ and consider the composite

$$G(m) \overset{\mathrm{Ad}}{\rightarrow} \mathrm{SO}(N) \overset{\text{diagonal}}{\rightarrow} \mathrm{SO}(kN + l) \rightarrow \mathrm{SU}(kN + l),$$

where the second map is given by

$$A \mapsto \begin{pmatrix} A & & & \\ & \ddots & & \\ & & A & \\ & & & I_l \end{pmatrix}$$

and the third map is the usual inclusion (complexification) map. When $kN + l \leqslant n$, we can restrict the usual transitive action on $W_{n,2}$ to $\mathrm{SU}(kN + l)$ and thus obtain an action of $G(m)$. Such actions of $G(m)$ are called *linear models* of $G(m)$ on $W_{n,2}$ *of adjoint type*. However, it is easy to see that the principal isotropy types of these linear models are always trivial.

The main result of this paper asserts that aside from mild dimension restrictions and assumptions on fixed point sets there can be no smooth action of $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, $\mathrm{SO}(m)$, or $\mathrm{Spin}(m)$ on $W_{n,2}$, n odd, with connected principal isotropy type (T).

MAIN THEOREM. *Let* $G(m) = \mathrm{SU}(m)$, $\mathrm{Sp}(m)$, $\mathrm{SO}(m)$, *or* $\mathrm{Spin}(m)$. *Then there can be no smooth action of* $G(m)$ *on* $W_{n,2}$, *n odd, with connected principal isotropy type* (T), *where T is a maximal torus of* $G(m)$, *if*

(i) $m \geqslant 5$; *and if* $G(m) = \mathrm{SO}(m)$ *or* $\mathrm{Spin}(m)$ *then we need also*

(ii) $F(T_2, X) \neq \varnothing$, T_2 *being a maximal* 2-*torus of* $\mathrm{SO}(m)$.[3]

From the proof of the main theorem, we immediately obtain the following:

MAIN COROLLARY. *Let* $G(m) = \mathrm{SU}(m)$ *or* $\mathrm{Sp}(m)$ *act smoothly on* $W_{n,2}$, *n odd, such that* $F(T, X) \neq \varnothing$. *Then if* $m \geqslant 5$ *the connected principal isotropy type* (H^0) *cannot be* $(\mathrm{SU}(2) \times \cdots \times \mathrm{SU}(2))$ ($[m/2]$-*factors*) *or* $(\mathrm{Sp}(1) \times \cdots \times \mathrm{Sp}(1))$ (*m-factors*) *respectively.*

REMARK. The orthogonal representations which give rise to the above principal isotropy types are $[\Lambda^2\mu_m + \text{trivial}]_{\mathbf{R}}$ and $[c\Lambda^2\nu_m + \text{trivial}]_{\mathbf{R}}$ respectively [**6**, **9**]. It is possible to define smooth actions on $S^{2n-3} \times S^{2n-1}$ using these representations to obtain the above principal isotropy types. However, for the linear models on $W_{n,2}$ defined in terms of these representations, the principal isotropy types are trivial.

The paper is organized as follows.

In §1, we make some remarks about the linear models of adjoint type and the strategy for the proof of the main theorem. In §§2 and 3, we prove the main theorem for the cases of $\mathrm{SU}(m)$ and $\mathrm{Sp}(m)$ respectively. In §4, we sketch a proof for the case of $\mathrm{SO}(m)$. In the last section we discuss the case of $\mathrm{Spin}(m)$ and the proof of the main corollary. We close the section with some remarks about the low dimensional cases.

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[3] See footnote 2.

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1. The linear models of adjoint type and the strategy for proving the main theorem. Consider the linear models of adjoint type defined by the representations $k\,\mathrm{Ad} \oplus$ trivial. The following are easily verified:

(1) The inequality $kN \leqslant n$ must hold if a nontrivial action exists.

(2) The principal isotropy subgroup is trivial.

(3) Assume $F(T, X) \neq \varnothing$, then the reduced geometrical weight system is $4k\Delta(G(m))$, where $\Delta(G(m))$ denotes the (reduced) root system of $G(m)$.

(4) If $G = \mathrm{SU}(m)$ and $F(G, X) \neq \varnothing$, then $F(T, X) = F(T_p, X)$ for p an odd prime. If in addition $m \geqslant 3$, $F(T, X) = F(T_2, X)$. If $G = \mathrm{Sp}(m)$,

$$F(T_2, X) \underset{\neq}{\supset} F(T, X) = F(T_p, X) \quad \text{for } p \text{ an odd prime.}$$

If $G = \mathrm{SO}(m)$,

$$F(T_2, X) \underset{\neq}{\subset} F(T, X) = F(T_p, X) \quad \text{for } p \text{ an odd prime.}$$

Furthermore, $F(G, X)$ is always a complex Stiefel manifold $W_{n-kN,2}$.

The proof of the main theorem will proceed by contradiction. Let $G(m)$ act smoothly on X such that the connected principal isotropy type $(H^0) = (T)$. We seek a contradiction to the topological splitting principle of Chang and Skjelbred [5]. Before we recall this, we state the following

DEFINITION. Let $G = T_p$ be a p-torus, where $p = 0, 2$, or an odd prime. Suppose G acts topologically on a mod p Poincaré Duality space X and suppose $F(G, X) \neq \varnothing$. Let F^i denote the ith connected component of $F(G, X)$. Then the *reduced local geometrical p-weight system at* F^i is $\Omega_p'^{(i)}(X) = \{w_k; n_{ki}\}$ where $w_k \in H^1(B_G; \mathbf{Z}/p)$ if $p = 2$ or an odd prime and $w_k \in H^2(B_G; \mathbf{Q})$ if $p = 0$, $w_k^\perp = \{g \in G \mid w_k(g) = 0\}$ if $p = 2$ or an odd prime and $w_k^\perp =$ the identity component of $\{g \in G \mid w_k(g) = 0\}$ if $p = 0$, $w_k^\perp$ are precisely the p-corank 1 (connected) isotropy subgroups of the G-action, and $n_{ki} = \dim[F(w_k^\perp, X)] - \dim[F^i(G, X)]$ if $p = 2$, $n_{ki} = \frac{1}{2}[\dim[F(w_k^\perp, X)] - \dim F^i(G, X)]$ if $p \neq 2$.

For the properties of p-weights, see [7, 9].

Topological splitting principle [5]. Notation as above, then F^i is again a mod p Poincaré Duality space. Furthermore, let $j_i^*\colon H_G^*(X; \mathbf{Z}/p) \to H_G^*(F^i; \mathbf{Z}/p)$ be the homomorphism in equivariant cohomology induced by the inclusion $F^i \subseteq X$. Let f^i be a mod p orientation class of F^i. Then the ideal $I_{f^i}(X, F^i) = \{a \in H^*(B_G; \mathbf{Z}/p)/af^i \in \operatorname{im} j_i^*\}$ is principal and is generated by $\prod_k w_k^{n_k}$.

We propose to calculate $I_{f^i}(X, F^i)$ in two different ways and show that the answers are incompatible with each other.

First we compute $\prod w_k^{n_k}$. To compute the w_k from the condition $(H^0) = (T)$, we need

THEOREM A8, C3, B5 (§§7, 8, 9 *in* [8]). *Let* $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, *or* $\mathrm{SO}(m)$ *act smoothly on* M, *a compact, connected manifold whose first two Pontrjagin classes vanish. If* $(H^0) = (T)$ *is a subtorus of rank* $\geqslant 2$, *then for every* x *in* M, G_x^0 *has the same rank as* T *and the local slice representation at* x *is given by* $\phi_x \mid G_x^0 = \mathrm{Ad}\, G_x^0 +$ *trivial.* □

We next compute I_{f^i} by studying the map j_i^*. For this, we need information regarding $H_G^*(X; \mathbf{Z}/p)$ and $H^*(F; \mathbf{Z}/p)$. The former amounts to studying the Serre spectral sequence of the fibration $X \to X_G \to B_G$; the latter comes from a theorem of Bredon-Skjelbred [**4, 9**] and a theorem of J. C. Su [**10**].

The comparison between the two calculations of I_{f^i} becomes the question whether an equation with variables in $H^*(B_G; \mathbf{Z}/p)$ and with Sq^2 as an operator has a solution or not. The presence of Sq^2 in the equation represents the input of the difference in homotopy type between $W_{n,2}$, n odd, and $S^{2n-3} \times S^{2n-1}$.

To show that the equation under consideration has no solution, it is easier first to restrict our attention to one local weight w_k at a time and then use the groups $N(T)/T$ or $N(T_2)/T_2$ to piece together the information obtained from considering each w_k. Hence in the actual proofs, the determination of invariance under $N(T)/T$ or $N(T_2)/T_2$ is of utmost importance.

Note that in [**12**] solving an equation such as that above led to information about the orbit structure and fixed point set structure of actions of regular type. In this paper, the insolvability of such an equation shows the impossibility of an action with $(H^0) = (T)$. We feel that the cohomology classes a and b in both papers should be regarded as equivariant characteristic classes of the particular action under study. Suitably applied, the topological splitting principle then embodies the topology of the manifold and the characteristics of the action in the form of a relation between these equivariant characteristic classes and cohomology operations.

2. Proof of the main theorem for $\mathrm{SU}(m)$. We begin with a series of lemmas.

LEMMA 1. *Let $F = F(T, X)$, $X = W_{n,2}$, n odd. Then F is a rational cohomology product of two odd spheres and j^*: $H_T^*(X, \mathbf{Q}) \to H_T^*(F; \mathbf{Q})$ is a monomorphism.*

PROOF. This follows from a standard argument. □

LEMMA 2. *If $m \geq 3$, then $F(T_2, X) = F(T, X) = F(T_p, X)$, for any odd prime p. Thus $F(T, X)$ is an integral cohomology product of two odd spheres.*

PROOF. We prove the first equality and leave the second to the reader. Note first that $F(T_2, X)$ and $F(T, X)$ are manifolds. Let C be the component of $F(T_2, X)$ containing $F(T, X)$. We claim that $C = F(T, X)$. Consider a G_x slice at any $x \in F(T, X)$. Now $G_x \supseteq T \supseteq T_2$. By Theorem A8 and Proposition 2 on p. 76 of [**9**], the reduced geometrical weight system is $\Delta(\mathrm{SU}(m))$. Upon restriction to T_2, no root goes to 0 if $m \geq 3$. Hence in a neighborhood of x, $\dim F(T, X) = \dim F(T_2, X)$ and the claim follows immediately.

Next we consider the Serre spectral sequence of $X_{T_2} \to B_{T_2}$ with $\mathbf{Z}/2$ coefficients. By elementary arguments, we see that X is totally nonhomologous to 0 in the fibration and by Corollary 2 on p. 46 of [**9**], $\dim_2 H^*(F(T_2, X); \mathbf{Z}/2) = \dim_2 H^*(X; \mathbf{Z}/2) = 4$. But $4 = \dim_2 H^*(F(T_2, X); \mathbf{Z}/2) \geq \dim_2 H^*(C; \mathbf{Z}/2) = \dim_2 H^*(F(T, X); \mathbf{Z}/2) \geq \dim_{\mathbf{Q}} H^*(F(T, X); \mathbf{Q}) = 4$. So equality holds throughout and $F(T_2, X)$ must be connected. □

LEMMA 3. *$H_T^*(X;\mathbf{Z}) \simeq \Lambda_R(\tilde{x}, \tilde{y})$, where $R = H^*(B_T;\mathbf{Z})$ and $\tilde{x}$, $\tilde{y}$ are lifts of generators of $H^*(X;\mathbf{Z})$. The same is true with $\mathbf{Q}$ or $\mathbf{Z}/2$ coefficients and we may assume that the generators correspond under tensoring with $\mathbf{Q}$ or reduction* mod 2 *respectively.*

PROOF. It is similar to proofs given in §4 of [**12**] and so will be omitted. □

We have a commutative diagram with the maps shown.

$$\begin{array}{ccc} H_T^*(X;\mathbf{Z}) & \xrightarrow{j^*} & H_T^*(F;\mathbf{Z}) \\ \downarrow \otimes \mathbf{Q} & & \downarrow \otimes \mathbf{Q} \\ H_T^*(X;\mathbf{Q}) & \xrightarrow{j^*} & H_T^*(F;\mathbf{Q}) \end{array}$$

Let $H^*(F;\mathbf{Z}) \approx \Lambda(f_1, f_2)$, then $j^*(\tilde{x}) = a \otimes f_1 + b \otimes f_2$, where $a, b \in H^*(B_T;\mathbf{Z})$. We can then reduce mod 2. By abuse of notation we let $j^*\colon H_T^*(X;\mathbf{Z}/2) \rightarrowtail H_T^*(F;\mathbf{Z}/2)$ send $\tilde{x}$ to $af_1 + bf_2$, with $a, b \in H^*(B_T;\mathbf{Z}/2)$. (That j^* is a monomorphism is established easily; see Lemma 2 in [**12**].) The ideal $I_{f_1 f_2}(X, F)_{(2)} = \{a \in H^*(B_T;\mathbf{Z}/2)/af_1 f_2 \in \operatorname{im} j^*\}$ is the reduction mod 2 of $I_{f_1 f_2}(X, F)$. Applying the topological splitting principle, we obtain the equations $a^2 + a\,\mathrm{Sq}^2 b + b\,\mathrm{Sq}^2 a = \prod_{i<j}(\theta_i - \theta_j)$ if $\mathrm{Sq}^2 f_1 = f_2$ and $a\,\mathrm{Sq}^2 b + b\,\mathrm{Sq}^2 a = \prod_{i<j}(\theta_i - \theta_j)$ if $\mathrm{Sq}^2 f_1 = 0$, since the left-hand sides of these equations generate the ideal $I_{f_1 f_2}(X, F)_{(2)}$, as can be seen from a routine computation.

Next we consider the invariance of a and b. There is no a priori reason why $F(T, X) = F(G, X)$ and so we may not conclude anything about the invariance of a and b by appealing to $H_G^*(X;\mathbf{Z}) \to H_G^*(F;\mathbf{Z})$, as was done in [**12**].

Let W be the Weyl group of $\mathrm{SU}(m)$ and $w \in W$. Then $w \cdot \tilde{x} = \tilde{x}$ because $\tilde{x}$ is the unique lift of $x \in H^{2n-3}(X;\mathbf{Z}/2)$ to $H_T^{2n-3}(X;\mathbf{Z}/2)$. We need to know how W acts on $H^*(F;\mathbf{Z}/2)$.

LEMMA 4. *If $m \geq 5$, the alternating subgroup $\mathcal{A}_m \subseteq W$ acts trivially on $H^*(F;\mathbf{Z}/2)$.*

PROOF. The problematic case occurs when the generators of $H^*(F;\mathbf{Z}/2)$ lie in the same dimension. In that case we have a representation of Σ_m on $\mathbf{Z}/2 \oplus \mathbf{Z}/2$, or a homomorphism $\Sigma_m \to \mathrm{GL}(\mathbf{Z}/2 \oplus \mathbf{Z}/2) \approx \Sigma_3$. The only nontrivial normal subgroup of Σ_m, $m \neq 4$, is $\mathcal{A}_m$. If $m \geq 5$, there must be a kernel which is either Σ_m of $\mathcal{A}_m$. □

COROLLARY. *If $m \geq 5$, a and b are invariant under the action of $\mathcal{A}_m \subseteq \Sigma_m$ on $H^*(B_T;\mathbf{Z}/2)$.* □

Let $\theta = \theta_i - \theta_j$, say $\theta_1 - \theta_2$ for convenience, and consider $F(\theta^\perp, X)$. In order to use the topological splitting principle, we want $\theta_1 - \theta_2$ to be the only root to vanish on the maximal p-tori of the subtorus $\theta^\perp$ of T. This is the case when $m \geq 5$, as can be easily verified. Then the proof of Lemma 2 shows that $F(\theta^\perp, X)$ is an integral cohomology product of two odd spheres.

PROOF OF MAIN THEOREM FOR SU(m). We have the following commutative diagram

$$\begin{array}{ccc} H_T^*(X;\mathbf{Z}/2) & \overset{j^*}{\rightarrowtail} & H_T^*(F;\mathbf{Z}/2) \\ & {\scriptstyle j_2^*}\searrow \quad\quad \nearrow{\scriptstyle j_1^*} & \\ & H_T^*(F';\mathbf{Z}/2) & \end{array}$$

where $F' = F(\theta^{\perp}, X)$, $H^*(F';\mathbf{Z}/2) = \Lambda(f_1', f_2')$, $\dim f_1' \leqslant \dim f_2'$, and $H^*(F;\mathbf{Z}/2) = \Lambda(f_1, f_2)$, $\dim f_1 \leqslant \dim f_2$. j^* is a monomorphism because of the commutative diagram

$$\begin{array}{ccc} H_T^*(X;\mathbf{Z}/2) & \to & H_T^*(F;\mathbf{Z}/2) \\ \downarrow & & \downarrow \\ H_{T_2}^*(X;\mathbf{Z}/2) & \rightarrowtail & H_{T_2}^*(F;\mathbf{Z}/2) \end{array}$$

in which the bottom map is injective since $F = F(T_2, X)$ and the usual arguments apply. Also, $H^*(B_T;\mathbf{Z}/2) \subseteq H^*(B_{T_2};\mathbf{Z}/2)$ shows that the left map is a monomorphism. Similarly, j_1^* is a monomorphism and $H_T^*(F',\mathbf{Z}/2) \approx \Lambda_{R_T}(\tilde{f}_1', \tilde{f}_2')$, where $\tilde{f}_i'$ is a lift of f_i'. Note that we do not know whether we can choose $\tilde{f}_2' = \mathrm{Sq}^2 \tilde{f}_1'$. So let $j_1^*(\tilde{f}_1') = e_1\theta^{c_1}f_1 + e_2\theta^{c_2}f_2, j_1^*(\tilde{f}_2') = g_1\theta^{d_1}f_1 + g_2\theta^{d_2}f_2$. Now

$$j_1^*(\tilde{f}_1'\tilde{f}_2') = (e_1g_2\theta^{c_1+d_2} + e_2g_1\theta^{d_1+c_2})f_1f_2,$$

so that by the topological splitting principle $\theta = e_1g_2\theta^{c_1+d_2} + e_2g_1\theta^{d_1+c_2}$. Clearly, either $e_2g_1 = 1, e_1g_2 = 0$ or $e_2g_1 = 0, e_1g_2 = 1$.

(a) $e_2g_1 = 1, e_1g_2 = 0, d_1 + c_2 = 1$.

If $d_1 = 1$, $c_2 = 0$, there are several possibilities. Suppose $\dim f_1' = \dim f_2'$, then $c_1 = 1$ and $\psi = \Pi_{i<j}(\theta_i - \theta_j)$ divides a. Consequently, neither of the two equations $a^2 + a\,\mathrm{Sq}^2 b + b\,\mathrm{Sq}^2 a = \psi$, $a\,\mathrm{Sq}^2 b + b\,\mathrm{Sq}^2 a = \psi$ can be satisfied. The other possibility is for $\dim f_1 = \dim f_2$. If $e_1 = 0$, then again ψ divides a and we get a contradiction. Hence $e_1 = 1$ and $g_2 = 0$. Let $j_2^*(\tilde{x}) = a_1\tilde{f}_1' + a_2\tilde{f}_2'$. Then $a_1 + \theta a_2 = a$ and $a_1 = b$. It follows that $a + b = \theta a_2$. If this is 0, $a = b$, and so $a^2 = \psi$, a contradiction. So $a + b \neq 0$, $\dim(a+b) \geqslant \dim\psi$. In particular, $\dim a \geqslant \dim \psi$; again none of the equations can be satisfied.

If $d_1 = 0$, $c_2 = 1$, then $d_2 \geqslant 1$ and so ψ divides b. We are then forced into the case $\mathrm{Sq}^2 f_1 = f_2$. But then $a^2 = \psi$, a contradiction.

(b) $e_2g_1 = 0, e_1g_2 = 1, c_1 + d_2 = 1$.

If $d_2 = 0$, $c_1 = 1$, then either $g_1 = 0$ or else $d_1 \geqslant 1$ so ψ divides a. Again this is impossible. If $d_2 = 1$, $c_1 = 0$, then either $\dim f_1 = \dim f_2$ or $\dim f_1 < \dim f_2$. In the first case if $e_2 = 0$, then ψ divides b. Hence only $a^2 + a\,\mathrm{Sq}^2 b + b\,\mathrm{Sq}^2 a = \psi$ holds. But a $\mathrm{Sq}^2 b$ and $b\,\mathrm{Sq}^2 a$ cannot be nonzero, so $a^2 = \psi$, which cannot hold. Hence $e_2 = 1$. By reversing f_1 and f_2, we may use the argument in (a) to complete the proof. If $\dim f_1 < \dim f_2$, then $e_2 = 0$ and so ψ divides b. We have just treated this situation.

The proof is complete. □

3. Proof of the main theorem for Sp(m). We now let Sp(m) act smoothly on $X = W_{n,2}$, n odd, with $(H^0) = (T)$ (hence $F(T_2, X) \neq \varnothing$). We let $\Omega_2^{(j)}(X)$ denote the reduced geometrical 2-weights system of the restricted T_2-action at F^j. Note that for smooth actions $\Omega_{T_2}^{(j)}(X)$ can also be described as the set of nonzero 2-weights of the slice representation at any $x \in F_j$. The n_{ki}'s are the (algebraic) multiplicities of the 2-weights.

LEMMA 1. *$\Omega_2^{(j)}(X)$ is invariant under the action of the alternating subgroup $\mathcal{A}_m \subseteq \Sigma_m \subseteq N(T_2)/T_2$ on $H^1(B_{T_2}; \mathbf{Z}/2)$ if $m \geqslant 5$.*

PROOF. By the theorem of J. C. Su [**10**], F consists at most of 4 components. Each element of $N(T_2)/T_2$ permutes these components. For a component C of F we study the isotropy subgroup of C under the action of $\Sigma_m \subseteq N(T_2)/T_2$. If this is Σ_m, then C is fixed by Σ_m and the usual proof of the invariance of $\Omega_2'(X)$ for a connected $F(T_2, X)$ applies (see, for example, Corollary 2 on p. 75 of [**9**]). If the isotropy subgroup H of C has index 2 in Σ_m, it must be $\mathcal{A}_m$ and the same argument shows the invariance of $\Omega_2^{(j)}(X)$ under $\mathcal{A}_m$. Suppose $[\Sigma_m : H] = 3$, then the orbit of C consists of three elements and we get a homomorphism $\phi\colon \Sigma_m \to \Sigma_3$ whose image contains at least two nonzero elements. Now Ker ϕ is normal in Σ_m and must have index 3 or 6, but no such subgroup exists. Lastly, assume $[\Sigma_m : H] = 4$. By the above reasoning, we get a homomorphism $\phi\colon \Sigma_m \to \Sigma_4$. Hence Ker ϕ has index $\geqslant 4$, which is impossible. □

By elementary arguments, we see that X is totally nonhomologous to 0 in the fibration $X_{T_2} \xrightarrow{p} B_{T_2}$; $\dim_2 H^*(F; \mathbf{Z}/2) = 4 = \dim_2 H^*(X; \mathbf{Z}/2)$; $j^*\colon H^*_{T_2}(X; \mathbf{Z}/2) \to H^*_{T_2}(F; \mathbf{Z}/2)$ is a monomorphism. Furthermore, if $\tilde{x}$ is the unique lift of the generator $x \in H^{2n-3}(X; \mathbf{Z}/2)$ such that the component of $j^*(\tilde{x})$ in $H^*_{T_2}(F_1; \mathbf{Z}/2)$ has no constant term and $\tilde{y} = \mathrm{Sq}^2\,\tilde{x}$, then 1, $\tilde{x}$, $\tilde{y}$, $\tilde{x}\tilde{y}$ form a free $H^*(B_{T_2}; \mathbf{Z}/2)$ base for $H^*_{T_2}(X; \mathbf{Z}/2)$. For more information see §6 of [**12**].

Next we would like to study how $N(T_2)/T_2$ acts on the domain and range of j^*.

LEMMA 2. *The alternating subgroup $\mathcal{A}_m \subseteq \Sigma_m \subseteq N(T_2)/T_2$ acts trivially on $H^*(F(T_2, X); \mathbf{Z}/2)$ if $m \geqslant 5$.*

PROOF. By the proof of Lemma 1, $\mathcal{A}_m$ fixes each component of F. In the cases where F consists of more than one component, it is obvious from Su's theorem that $\mathcal{A}_m$ acts trivially on $H^*(F; \mathbf{Z}/2)$. If there is only one component, then we may have a representation of Σ_m on $\mathbf{Z}/2 \oplus \mathbf{Z}/2$, i.e., a homomorphism

$$\phi\colon \Sigma_m \to \Sigma_3 \approx \mathrm{GL}(\mathbf{Z}/2 \oplus \mathbf{Z}/2).$$

If $m \geqslant 5$, Ker $\phi = \Sigma_m$ or $\mathcal{A}_m$ and thus our lemma follows. □

Now let $n \in N(T_2)/T_2$, then $j^*(n^*\tilde{x}) = n^*j^*(\tilde{x})$. Since the constant of the component of $j^*(\tilde{x})$ in $H^*_{T_2}(F_1; \mathbf{Z}/2)$ is chosen to be 0, that of the component of $j^*(n^*\tilde{x})$ in $H^*_{T_2}(F_1; \mathbf{Z}/2)$ is also 0. But it follows from this that $n^*\tilde{x} = \tilde{x}$. In view of Lemma 2, if $m \geqslant 5$, then the coefficients in $j^*(\tilde{x})$ are polynomials invariant under $\mathcal{A}_m$.

At this point we explain the technical difficulties encountered in proving the main theorem for Sp(m). The group Sp(m) has $\pm 2x_i$ as some of its roots. In the adjoint representation, therefore, $F(T_2, X) \supsetneqq F(T, X)$. When we consider general smooth actions of $\mathrm{Sp}(m)$ on $W_{n,2}$, n odd, with $(H^0) = (T)$, we no longer have $F(T_2, X) = F(T, X)$ and the arguments of the previous section fail. Because the operation Sq^2 is crucial to us, it is natural to work with T_2 instead of T. But the $\mathbf{Z}/2$-cohomology type of $F(T_2, X)$ has a priori numerous possibilities [**10**]. Even if $F(T_2, X)$ is again a $\mathbf{Z}/2$ cohomology product of two spheres, we need to know the parity of their dimensions and the difference between their dimensions.

To overcome these difficulties, we observe that

LEMMA 3. *Let* $\mathrm{Sp}(m)$ *act smoothly on* X *with* $(H^0) = (T)$, *then the action factors through* $\mathrm{center}(\mathrm{Sp}(m)) \approx \mathbf{Z}/2$.

PROOF. $\mathbf{Z}/2 = \mathrm{center}(\mathrm{Sp}(m)) \subseteq T$ and every point $x \in X$ has $G_x^0 \supseteq gTg^{-1}$. Therefore, $\mathbf{Z}/2 \subseteq G_x^0$. □

We shall consider the induced action of the adjoint group of $\mathrm{Sp}(m)$ on X. This makes the determination of $H^*(F(T_2, X); \mathbf{Z}/2)$ easier.

Consider the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & T & \xrightarrow{i} & N(T) & \xrightarrow{\pi} & W(\mathrm{Sp}(m)) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow\approx & & \\ 1 & \to & \bar{T} & \xrightarrow{\bar{i}} & N(\bar{T}) & \xrightarrow{\bar{\pi}} & W(\mathrm{Sp}(m)/\mathbf{Z}/2) & \to & 1 \\ & & 1 & & 1 & & & & \end{array}$$

where $\bar{T} = T/\mathbf{Z}/2$. Recall that $W(\mathrm{Sp}(m)) \approx \Sigma_m \times_\rho (\mathbf{Z}/2)^m$, where $\rho\colon \Sigma_m \to \mathrm{Aut}((\mathbf{Z}/2)^m)$ is defined by permutation in the obvious manner.

LEMMA 4. *The* 2-*rank of* $\mathrm{Sp}(m)/(\mathbf{Z}/2)$ *is at least* $m + 1$.

PROOF. Let

$$\tau_i = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \qquad 1 \leqslant i \leqslant m - 1,$$

$$\delta = \begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix}, \qquad \lambda = \begin{pmatrix} j & & \\ & \ddots & \\ & & j \end{pmatrix}.$$

Because these elements commute up to -1 and their squares are either $+1$ or -1, their images in $\mathrm{Sp}(m)/(\mathbf{Z}/2)$ form a $\mathbf{Z}/2$ torus of rank $m + 1$. □

LEMMA 5. *Let* Sp(m) *act smoothly on* $W_{n,2}$ *such that* $F(T, X) \neq \varnothing$ *and* $\Omega'_T(X) = \Delta(\mathrm{Sp}(m))$ *or* $\{\pm(\theta_i - \theta_j);\ \pm(\theta_i + \theta_j)\}$. *Then the action factors through* $\mathbf{Z}/2 = \mathrm{center}(\mathrm{Sp}(m))$ *and* $F(T, X) = F(\bar{T}, X) = F(Q_2, X)$. *Hence it is a* $\mathbf{Z}/2$ *cohomology product of two odd spheres. Here* Q_2 *is the* 2-*torus in* $\mathrm{Sp}(m)/(\mathbf{Z}/2)$ *generated by* $\bar{\tau}_i$ *and* $\bar{\delta}$.

PROOF. First $Q_2 \subseteq \bar{T}$ and so $F(\bar{T}, X) \subseteq F(Q_2, X)$. Next let us evaluate the weights on Q_2. τ_i is represented in the Lie algebra of T by $(0,\dots,0,\frac{1}{2},0,\dots,0)$, with $\frac{1}{2}$in the ith place, and δ by $(\frac{1}{4},\dots,\frac{1}{4})$. Upon evaluation, one observes that no weight vanishes on Q_2 completely. We may now use the argument in Lemma 2-2 to conclude the proof. □

Let $t_1,\dots,t_m$ be variables dual to $\bar{\tau}_1,\dots,\bar{\tau}_{m-1}, \bar{\delta}$ respectively. Then a simple calculation shows that $\pm 2\theta_i$ corresponds to t_m; $\pm(\theta_i - \theta_j)$, $i<j<m$, corresponds to $t_i + t_j$; $\pm(\theta_i - \theta_m)$ corresponds to t_i; $\pm(\theta_i + \theta_j)$, $i<j<m$, corresponds to $t_i + t_j + t_m$; $\pm(\theta_i + \theta_m)$ to $t_i + t_m$.

LEMMA 6. *Assumptions as in Lemma* 5, *then*

$$\Omega'_{Q_2}(X) = 2m\{t_m\} \cup 2\{t_i, t_i + t_m, 1 \leqslant i \leqslant m-1; t_i + t_j, t_i + t_j + t_m, 1 \leqslant i < j < m\}$$

and $2\{t_i, t_i + t_m,\ 1 \leqslant i \leqslant m-1;\ t_i + t_j,\ t_i + t_j + t_m,\ 1 \leqslant i<j<m\}$ *respectively*. □

Let $t = t_1$, then $t_1^\perp$ is the 2-torus generated by $\bar{\tau}_2,\dots,\bar{\tau}_{m-1}, \bar{\delta}$. We see that $t_1^\perp \subset \overline{(\theta_1 - \theta_m)^\perp}$, where the bar indicates the image in $\bar{T}$. By the reasoning in Lemma 5, $F((\theta_1 - \theta_m)^\perp, X) = F(t_1^\perp, X)$ is a $\mathbf{Z}/2$ cohomology product of two odd spheres. We have the following commutative diagram:

$$\text{(I)}\qquad \begin{array}{ccc} \tilde{x} \in H^*_{Q_2}(X;\mathbf{Z}/2) & \xrightarrow{j^*} & H^*_{Q_2}(F(Q_2, X);\mathbf{Z}/2) \ni af_1 + bf_2 + cf_1f_2 \\ & j_2^* \searrow \quad \nearrow j_1^* & \\ & H^*_{Q_2}(F(t^\perp, X);\mathbf{Z}/2) & \end{array}$$

In the above, we assume the usual choices of $\tilde{x}$, $\tilde{y}$, $H^*(F(Q_2, X);\mathbf{Z}/2) \approx \Lambda(f_1, f_2)$, $\dim f_1 \leqslant \dim f_2$, and $H^*(F(t_1^\perp, X);\mathbf{Z}/2) \approx \Lambda(f_1'', f_2'')$, $\dim f_1'' \leqslant \dim f_2''$.

LEMMA 7. *The collection of* 2-*weights* $\{t_i, t_i + t_m,\ 1 \leqslant i \leqslant m-1;\ t_i + t_j,\ t_i + t_j + t_m,\ 1 \leqslant i<j<m\}$ *form a single orbit under the action of* $N(Q_2)/Q_2$.

PROOF. Consider the elements

$$E_i = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & I_{m-i-1} \end{pmatrix}, \qquad 1 \leqslant i \leqslant m-1,$$

and

$$J_i = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & j & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

with j in the ith position. Then one verifies that

$$\bar{E}_j \bar{\tau}_j \bar{E}_j^{-1} = \begin{cases} \bar{\tau}_i & \text{if } i < j, \\ \bar{\tau}_{i+1} & \text{if } i = j, \\ \bar{\tau}_i & \text{if } i > j+1, \\ \bar{\tau}_{i-1} & \text{if } i = j+1, \end{cases}$$

$\bar{E}_j \bar{\delta} \bar{E}_j^{-1} = \bar{\delta}$, $\bar{J}_i \bar{\tau}_k \bar{J}_i^{-1} = \bar{\tau}_k$, $1 \leq k \leq m-1$, $\bar{J}_i \bar{\delta} \bar{J}_i^{-1} = \bar{\tau}_i \bar{\delta}$. Note that $\bar{\tau}_m = \bar{\tau}_1 \ldots \bar{\tau}_{m-1}$. We therefore see that if $1 \leq i \leq m-2$, $\bar{E}_i^{-1}$ interchanges t_i and t_{i+1} and leaves all other t_j's fixed. $\bar{E}_{m-1}^{-1}$ takes t_j to $t_j + t_{m-1}$ if $j \leq m-2$ and fixes t_{m-1} and t_m. As for $\bar{J}_i^{-1}$, it sends t_i to $t_i + t_m$ and fixes all t_j, $j \neq i$, $1 \leq i \leq m-1$; $\bar{J}_m^{-1} t_i = t_i + t_m$, $1 \leq i \leq m-1$, and $\bar{J}_m^{-1} t_m = t_m$. The claim of the lemma follows immediately from the above observations. □

LEMMA 8. *The coefficients a, b, c are invariant under the action of* E_i^{-1}, $1 \leq i \leq m-1$, *and under some element of* $N(Q_2)/Q_2$ *that takes* t_1 *to* $t_1 + t_m$.

PROOF. This is certainly the case when $\dim f_1 \neq \dim f_2$ since $N(Q_2)/Q_2$ acts trivially on $H^*(F(Q_2, X); \mathbf{Z}/2)$.

First observe that for the desired invariance to hold, all we need is that E_i^{-1}, $1 \leq i \leq m-1$, should act trivially on $H^*(F(Q_2, X); \mathbf{Z}/2)$ and that some element of $N(Q_2)/Q_2$ which takes t_1 to $t_1 + t_m$ acts trivially on $H^*(F(Q_2, X); \mathbf{Z}/2)$. Because $m \geq 5$, we observe that we can just as well require products $\bar{E}_i^{-1} \bar{E}_j^{-1}$ to act trivially on $H^*(F(Q_2, X); \mathbf{Z}/2)$. Thirdly, notice that the action of $E_i^{-1} E_j^{-1}$ on $H^*(F(Q_2, X); \mathbf{Z}/2)$ is the same as that of $\bar{E}_i^{-1} \bar{E}_j^{-1}$ on $H^*(F(Q_2, X); \mathbf{Z}/2)$ since $F(T, X) = F(Q_2, X)$.

But the products $E_i^{-1} E_j^{-1}$ lie in the alternating subgroup $\mathcal{Q}_m \subseteq N(T)/T$. $\mathcal{Q}_m$ for $m \geq 5$ must act trivially on $H^*(F(Q_2, X); \mathbf{Z}/2)$. Next consider the element

$$E = E_1 E_3 J_1 E_1 E_3 J_1^{-1} = \begin{pmatrix} j & 0 & 0 & 0 & \\ 0 & -j & 0 & 0 & \\ 0 & 0 & -1 & 0 & \\ 0 & 0 & 0 & -1 & \\ & & & & I \end{pmatrix}.$$

$\overline{E_1 E_3}$ lies in $N(Q_2)/Q_2$ and so does $\bar{J}_1$. $\overline{E_1 E_3}$ lies in the normal subgroup of $N(Q_2)/Q_2$ which is the kernel of the action on $H^*(F(Q_2, X); \mathbf{Z}/2)$. Therefore, $\bar{E}$ also acts trivially on $H^*(F(Q_2, X); \mathbf{Z}/2)$. But a simple calculation shows that $\bar{E} \bar{\tau}_i \bar{E}^{-1} = \bar{\tau}_i$, $1 \leq i \leq m-1$, $\bar{E} \bar{\delta} \bar{E}^{-1} = \bar{\tau}_1 \bar{\tau}_2 \bar{\delta}$. Hence $\bar{E}^{-1}$ takes t_1 to $t_1 + t_m$ and acts trivially on $H^*(F(Q_2, X); \mathbf{Z}/2)$. We have therefore shown the required invariance of a and b. $\square$

LEMMA 9. *$F(t_m^\perp, X) = F(T_2, X)$ is a $\mathbf{Z}/2$ cohomology product of 2 spheres.*

PROOF. This follows from the fact that $F(Q_2, X)$ is a $\mathbf{Z}/2$ cohomology product of two odd spheres: $H^*_{Q_2}(X; \mathbf{Z}/2)$ is an exterior algebra on $\tilde{x}, \tilde{y}$, $H^*_{t_m^\perp}(X; \mathbf{Z}/2)$ is then an exterior algebra on $\tilde{x}, \tilde{y}$, and by Borel's localization theorem (Proposition 2, p. 39 of [9]) $H^*(F(t_m^\perp, X); \mathbf{Z}/2)$ must be an exterior algebra on two generators. It is clear that $F(t_m^\perp, X) = F(\bar{T}_2, X) = F(T_2, X)$. $\square$

We have therefore removed the first source of difficulties in the proof of the main theorem. We may now compute $\Omega_2'(X)$ by looking at a slice at $x \in F(T, X)$. By Proposition 2 on p. 76 of [9] and Theorem C3, we see that $\Omega_2'(X) = \Delta_2(\mathrm{Sp}(m)) = 4\{t_i + t_j,\ i < j\}$, where t_i now denotes the element of $H^1(B_{T_2}; \mathbf{Z}/2)$ dual to $\mathrm{diag}(1, \ldots, 1, -1, 1, \ldots, 1)$ (with -1 in the ith place).

LEMMA 10. *Suppose* Sp(m) *acts smoothly on* X *with* $(H^0) = (T)$ *and* $m \geq 5$. *If* $H^*(F(T_2, X); \mathbf{Z}/2) \approx \Lambda_{\mathbf{Z}/2}(f_1', f_2')$, *then* $\dim f_2' = 2 + \dim f_1'$.

PROOF. Let $t = t_1 + t_2$. We have a commutative diagram

$$\begin{array}{ccc} H^*_{T_2}(X; \mathbf{Z}/2) & \overset{j^*}{\rightarrowtail} & H^*_{T_2}(F(T_2, X); \mathbf{Z}/2) \\ & {\scriptstyle j_2^*}\searrow \quad \nearrow {\scriptstyle j_1^*} & \\ & H^*_{T_2}(F(t^\perp, X); \mathbf{Z}/2) & \end{array}$$

as before. $H^*(F(t^\perp, X); \mathbf{Z}/2) \approx \Lambda(f_1'', f_2'')$, $\dim f_1'' \leq \dim f_2''$, because by assumption $H^*(F(T_2, X); \mathbf{Z}/2)$ is an exterior algebra on two generators and hence $H^*_{T_2}(X; \mathbf{Z}/2) \approx \Lambda_{R_{T_2}}(\tilde{x}, \tilde{y})$; as a result $H^*_{t^\perp}(X; \mathbf{Z}/2)$ is also an exterior algebra on $\tilde{x}, \tilde{y}$ whence by the localization theorem of Borel, $H^*(F(t^\perp, X); \mathbf{Z}/2)$ must be an exterior algebra on two generators.

Because $\dim_2 H^*(F(t^\perp, X); \mathbf{Z}/2) = \dim_2 H^*(F(T_2, X); \mathbf{Z}/2)$, by Theorem VII-1-6 of [3] we see that the Serre spectral sequence of the fibration $F(t^\perp, X)_{T_2} \to B_{T_2}$ has simple coefficients and collapses. Let $\tilde{f}_1''$ and $\tilde{f}_2''$ be lifts of f_1'' and f_2'' such that $j_1^*(\tilde{f}_1'')$ and $j_1^*(\tilde{f}_2'')$ do not have constant terms. Then $1, \tilde{f}_1'', \tilde{f}_2'', \tilde{f}_1''\tilde{f}_2''$ form a basis of $H^*_{T_2}(F(t^\perp, X); \mathbf{Z}/2)$ over $H^*(B_{T_2}; \mathbf{Z}/2)$. Let $\theta = \prod_{i<j}(t_i + t_j)$.

Let $j_1^*(\tilde{f}_1'') = e_1 t^{c_1} f_1' + e_2 t^{c_2} f_2' + e_3 t^{c_3} f_1' f_2'$ and $j_1^*(\tilde{f}_2'') = g_1 t^{d_1} f_1' + g_2 t^{d_2} f_2' + g_3 t^{d_3} f_1' f_2'$. $j_1^*(\tilde{f}_1''\tilde{f}_2'') = t^4 f_1' f_2'$ and so by the topological splitting principle, $t^4 = e_1 g_2 t^{c_1 + d_2} + g_1 e_2 t^{c_2 + d_1}$. Now $c_1 + d_2 = 4$ or $c_2 + d_1 = 4$ and $e_1 g_2 + g_1 e_2 = 1$. Let $j^*(\tilde{x}) = \alpha_1 f_1' + \alpha_2 f_1' + \alpha_3 f_1' f_2'$. Recall that $\dim F \equiv 0(4)$ $(F = F(T_2, X))$. Hence $\mathrm{Sq}^1 f_1' = f_2'$ is impossible. Again applying the topological splitting principle, we see that if $\mathrm{Sq}^2 f_1' = 0$, then $\alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta^4$ and if $\mathrm{Sq}^2 f_1' = f_2'$, then $\alpha_1^2 + \alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta^4$. Finally, let $j_2^*(\tilde{x}) = a_0 + a_1 \tilde{f}_1'' + a_2 \tilde{f}_2'' + a_3 \tilde{f}_1''\tilde{f}_2''$. Then

$$\alpha_1 = a_1 e_1 t^{c_1} + a_2 g_1 t^{d_1},$$
$$\alpha_2 = a_1 e_2 t^{c_2} + a_2 g_2 t^{d_2}, \quad \text{and}$$
$$\alpha_3 = a_1 e_3 t^{c_3} + a_2 g_3 t^{d_3} + a_3 t^4.$$

Note that $c_1 \leq d_1$, $c_2 \leq d_2$.

Let us next consider the possibilities of $\mathrm{Sq}^2 \tilde{f}_1''$. Suppose $\mathrm{Sq}^2 \tilde{f}_1'' \neq 0$, then in terms of the additive structure of $H^*_{T_2}(F(t^\perp, X); \mathbf{Z}/2)$, $\mathrm{Sq}^2 \tilde{f}_1'' = e(t^r \otimes f_2'') + g(t^2 \otimes f_1'')$. If $g = 1$ and $e = 0$, then $\mathrm{Sq}^2 \tilde{f}_1'' = t^2 \tilde{f}_1''$. Otherwise, $e = 1$ and there are two cases. Note that since $\dim F \equiv 0(4)$, $\dim F' \equiv 0(4)$ ($F' = F(t^\perp, X)$) as well and $r = 0$ or 2. If $r = 2$, then $\dim f_1'' = \dim f_2''$; if $r = 0$, $\mathrm{Sq}^2 \tilde{f}_1''$ is a lift of f_2'' and so $\mathrm{Sq}^2 f_1'' = f_2''$. In the second case, we may assume that $\tilde{f}_2'' = \mathrm{Sq}^2 \tilde{f}_1''$.

Suppose $e_3 = 1$, then $\dim f_2'' \leqslant 4$ and so $\dim F' \leqslant 8$. Since $\dim F > 0$ and $\dim F \equiv 0(4)$, $\dim F = 4$.

A. $e_1 g_2 = 1, e_2 g_1 = 0$

(i) $c_1 = 4, d_2 = 0$. Because $d_1 \geqslant c_1 = 4$, θ^4 divides α_1, which cannot happen.

(ii) $c_1 = 3, d_2 = 1$. Clearly, $d_1 \geqslant c_1 = 3$ and θ^3 divides α_1. It follows that $\mathrm{Sq}^2 f_1' = 0$. If $e_2 = 0$, then θ divides α_2 and $\alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta^4$ cannot hold for dimension reasons. Thus $e_2 = 1$, $g_1 = 0$, and $\dim f_2' = \dim f_1' + 2$ is forced.

(iii) $c_1 = 0, d_2 = 4$. First suppose $e_2 = 0$, then θ^4 divides α_2. Hence only the equation $\alpha_1^2 + \alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta^4$ can hold. But $\dim \alpha_1 = 2 \dim \theta \geqslant \dim \alpha_2$ if $\alpha_2 \neq 0$. Consequently, $\alpha_2 = 0$. If $e_3 = 1$, then $\dim f_2'' \leqslant 4$, which cannot hold since $d_2 = 4$. Thus $j_1^*(\tilde{f}_1'') = \tilde{f}_1'$ and $j_1^*(\mathrm{Sq}^2 \tilde{f}_1'') = \tilde{f}_2' \neq 0$. Because $\dim f_2'' - \dim f_1'' \geqslant 4$, and $\mathrm{Sq}^2 \tilde{f}_1'' \neq t^2 \tilde{f}_1''$, we obtain a contradiction.

Hence $e_2 = 1$. This forces $\dim f_1' = \dim f_2'$ and $j_1^*(\tilde{f}_1'') = f_1' + f_2'$. Now $\alpha_1 = a_1$ and $\alpha_2 = a_1 + a_2 t^4$. $\alpha_1 \neq 0$ and $\alpha_1 + \alpha_2 = a_2 t^4 \neq 0$ (otherwise $\alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = 0$). So $\dim \alpha_1 = \dim(\alpha_1 + \alpha_2) \geqslant 4 \dim \theta$, a contradiction.

(iv) $c_1 = 1, d_2 = 3$. If $e_2 = 0$, then θ^3 divides α_2. Since $d_1 \geqslant c_1$, θ divides α_1. Consequently, only the equation $\alpha_1^2 + \alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta^4$ holds, i.e., $\mathrm{Sq}^2 f_1' = f_2'$. It follows that $\dim f_2' - \dim f_1' = 2$.

So let $e_2 = 1, g_1 = 0$. Now $\alpha_1 = a_1 t$ and so θ divides α_1. Also, $\dim f_1' = \dim f_2'$ is forced so that $\alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta^4$. Clearly $e_3 = 0$ and $c_2 = 1$. $\alpha_2 = a_1 t + a_2 t^3 = \alpha_1 + a_2 t^3$. Because $\alpha_2 \neq 0$ and $\alpha_1 + \alpha_2 \neq 0$, $\dim \alpha_2 = \dim(\alpha_1 + \alpha_2) \geqslant 3 \dim \theta$; together with θ dividing α_1, we get a contradiction.

(v) $c_1 = 2 = d_2$. Observe that $d_1 \geqslant c_1 = 2$ and so θ^2 divides α_1. Let $e_2 = 0$. Then θ^2 divides α_2 and only $\alpha_1^2 + \alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta^4$ can hold. But then $\mathrm{Sq}^2 f_1' = f_2'$ and so $\dim f_2' - \dim f_1' = 2$.

If $e_2 = 1$, then $g_1 = 0$ and θ^2 divides α_1. $\alpha_2 = a_2 t^2 + a_1 t^{c_2}$. If $c_2 = 2$, then $\dim f_1 = \dim f_2$ and θ^2 divides α_2. In this case, $\mathrm{Sq}^2 f_1' = 0$ but $\alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta^4$ cannot be satisfied. Hence $c_2 = 0$. We have $j_1^*(\tilde{f}_1'') = t^2 f_1' + f_2' + e_3 t^{c_3} f_1 f_2$, and thus $\dim f_2' = \dim f_1' + 2$.

B. $e_1 g_2 = 0, e_2 g_1 = 0$. Again arguments similar to those in A show that we must have $\dim f_2' = \dim f_1' + 2$. □

Proof of the Main Theorem for $\mathrm{Sp}(m)$. Let

$$\psi = \prod_{i=1}^{m} t_i(t_i + t_m) \prod_{1 \leqslant i < j \leqslant m-1} (t_i + t_j)(t_i + t_j + t_m).$$

By the topological splitting principle applied to the map $j^*\colon H^*_{Q_2}(X; \mathbf{Z}/2) \to H^*_{Q_2}(F(Q_2, X); \mathbf{Z}/2)$ we see that if $\mathrm{Sq}^2 f_1 = f_2$, then $a^2 + a \mathrm{Sq}^2 b + b \mathrm{Sq}^2 a = t_m^{2m} \psi^2$ and if $\mathrm{Sq}^2 f_1 = 0$, then $a \mathrm{Sq}^2 b + b \mathrm{Sq}^2 a = t_m^{2m} \psi^2$. Note that the above depends on

Lemma 6 and the knowledge of the $\mathbf{Z}/2$ cohomology of $F(Q_2, X)$ (including the parity of the dimensions of the generators). Note also that by Lemmas 7 and 8, a, b and ψ have suitable invariance properties under the action of some subgroup of $N(Q_2)/Q_2$. Finally, we observe that when m is even, $\dim F(Q_2, X) = 4n - 4 - 2m^2$ so that $\dim f_2 - \dim f_1 \geqslant 2$. When m is odd, then either $\dim f_1 = \dim f_2$ or else $\dim f_2 - \dim f_1 \geqslant 4$.

We apply the topological splitting principle now to the map

$$j_1^*\colon H^*_{Q_2}(F(t_1^\perp, X); \mathbf{Z}/2) \to H^*_{Q_2}(F(Q_2, X); \mathbf{Z}/2).$$

This is possible because the $\mathbf{Z}/2$ cohomology types of $F(t_1^\perp, X)$ and $F(Q_2, X)$ are known. By arguments similar to those in the proof of Lemma 10, we conclude that $\mathrm{Sq}^2 f_1 = 0$ and t_1^2 must divide a. By the invariance properties, ψ^2 divides a. Hence $\dim a \geqslant 2m(m-1)$, $\dim b \leqslant 2(m-1)$ and $\dim f_2 - \dim f_1 = \dim a - \dim b$.

Next we apply the topological splitting principle to the map

$$j_3^*\colon H^*_{Q_2}(F(t_m^\perp, X); \mathbf{Z}/2) \to H^*_{Q_2}(F(Q_2, X); \mathbf{Z}/2).$$

By Lemmas 9 and 10, $F(t_m^\perp, X) = F(T_2, X)$ has $\mathbf{Z}/2$ cohomology $\Lambda_{\mathbf{Z}/2}(f_1', f_2')$ where $\dim f_2' - \dim f_1' = 2$. Let $j_3^*(\tilde{f}_1') = e_1 t_m^{c_1} f_1 + e_2 t_m^{c_2} f_2 + e_3 t_m^{c_3} f_1 f_2$ and $j_3^*(\tilde{f}_2') = h_1 t_m^{d_1} f_1 + h_2 t_m^{d_2} f_2 + h_3 t_m^{d_3} f_1 f_2$. The splitting principle yields $t_m^{2m} = e_1 h_2 t_m^{c_1 + d_2} + e_2 h_1 t_m^{d_1 + c_2}$. There are two cases:

Case (1). $e_1 h_2 = 1$, $e_2 h_1 = 0$, $c_1 + d_2 = 2m$.

$\dim \tilde{f}_2' - \dim \tilde{f}_1' = 2 = d_2 + \dim f_2 - c_1 - \dim f_1 = d_2 - c_1 + \dim a - \dim b$. Since $\dim a \geqslant 2m(m-1) \geqslant 2(m-1) \geqslant \dim b$ for $m \geqslant 1$, $2 \geqslant d_2 - c_1 + 2m(m-1) - 2(m-1) = d_2 - c_1 + 2(m-1)^2$. So $2m \geqslant c_1 - d_2 \geqslant 2(m-1)^2 - 2 > 0$ if $m > 2$. We get $2(m+1) \geqslant 2(m-1)^2$ or $m + 1 \geqslant (m-1)^2$, contradicting $m \geqslant 5$.

Case (2). $e_1 h_2 = 0$, $e_2 h_1 = 1$, $d_1 + c_2 = 2m$.

Then $\dim \tilde{f}_2' - \dim \tilde{f}_2' = 2 = d_1 + \dim f_1 - c_2 - \dim f_2$. $2 = d_1 - c_2 + \dim b - \dim a$. Now $\dim a - \dim b = d_1 - c_2 - 2 \geqslant 2(m-1)^2$ as in (1). Hence $2m \geqslant d_1 - c_2 \geqslant 2(m-1)^2 + 2$ or $1 \geqslant m - 1$, again contradicting $m \geqslant 5 > 2$.

The proof is therefore complete. □

4. Proof of the main theorem for $\mathrm{SO}(m)$. We shall only sketch a proof for this case because the general techniques have been illustrated in detail in the previous two sections.

First we need to establish invariance properties. If T_2 denotes the standard maximal 2-torus in $\mathrm{SO}(m)$, then $N(T_2)/T_2 \approx \Sigma_m$.

LEMMA 1. *$\Omega_2^{(j)}(X)$ is invariant under the action of $\mathcal{Q}_m \subseteq \Sigma_m$ on $H^1(B_{T_2}; \mathbf{Z}/2)$ if $m \geqslant 5$.*

LEMMA 2. *$\mathcal{Q}_m \subseteq \Sigma_m$ acts trivially on $H^*(F(T_2, X); \mathbf{Z}/2)$ if $m \geqslant 5$.*

The proofs of these lemmas are the same as those for Lemmas 3-1 and 3-2.

LEMMA 3. *$\Omega_2^{(j)}(X)$ does not depend on j and all connected components of $F(T_2, X)$ have the same dimension.*

PROOF. For $x \in F_j(T_2, X)$, $\Omega_{T_2}^{(j)} = [\Delta_2(G) - \Delta_2(G_x)] + \Omega_2(S_x)$, where S_x is a slice at x. In Theorem B5, only $\varphi_x \mid G_x^0$ is known. Since T_2 may not be completely contained in G_x^0, we have to consider two cases.

If $\Delta_2(G) - \Delta_2(G_x) = \varnothing$, then since $\Delta_2(G_x) \subseteq \Delta_2(G)$ (see Theorem 2 in **[2]**) we have $\Delta_2(G) = \Delta_2(G_x)$. It follows that $G_x = G$. In that case, $\Omega_{T_2}^j(X) = \Omega_2(S_x)$ and G acts orthogonally on S_x with $(H^0) = (T)$. Hence $\Omega_2'(S_x) = \Delta_2(G)$.

If $\Delta_2(G) - \Delta_2(G_x)$ is nonempty, $\Omega_2^j(X)$ contains $\Delta_2(G)$ and so $\Omega_2'(S_x) \supseteq \Delta_2(G_x)$. We claim that equality holds. Let $t \in \Omega_2'(S_x) - \Delta_2(G_x)$. The entire orbit of t under $\mathcal{C}_m$ lies in $\Omega_2'(S_x)$ because $\Delta_2(G)$ and $\Omega_2^j(X)$ are invariant under $\mathcal{C}_m$. We may therefore assume $t = t_1 + \cdots + t_l$, $l \leqslant [m/2]$. The common kernel of $\{w \cdot t\}$, $w \in \mathcal{C}_m$, is 0. So some $w \cdot t$ will be nonzero when restricted to $T_2 \cap T$. This is a contradiction because $\Omega_2^j(X)|_{T_2 \cap T} = \Delta_2(G)|_{T_2 \cap T}$ by Theorem B5.

Hence in any case, $\Omega_2^j(X)' = \Delta_2(G)$. Note that the above proof depends heavily on the fact that the 2-roots in $\Delta_2(G)$ have multiplicity one and form a single orbit under the action of $\mathcal{C}_m$. Also, the fact $F(T_2 \cap T, X) = F(T, X)$ plays a role. (This fact is established using an argument similar to the one used in the proof of Lemma 2-2.) □

As in the Sp(m) case, X is totally nonhomologous to 0 in the fibration $X_{T_2} \to B_{T_2}$ and $\dim_2 H^*(X; \mathbf{Z}/2) = 4 = \dim_2 H^*(F; \mathbf{Z}/2)$. $j^*\colon H_{T_2}^*(X; \mathbf{Z}/2) \to H_{T_2}^*(F; \mathbf{Z}/2)$ is a monomorphism. Let $\tilde{x}$ be the unique lift of $x \in H^{2n-3}(X; \mathbf{Z}/2)$ for which the component of $j^*(\tilde{x})$ in $H_{T_2}^*(F_1; \mathbf{Z}/2)$ has no constant term. Set $\tilde{y} = \mathrm{Sq}^2\, \tilde{x}$. Although we do not know the product structure in $H_{T_2}^*(X; \mathbf{Z}/2)$, it is a free $H^*(B_{T_2}; \mathbf{Z}/2)$-module on $1, \tilde{x}, \tilde{y}, \tilde{x}\tilde{y}$.

Let t be a 2-root of G and consider $F' = F(t^\perp, X)$. Since $t^\perp$ is a 2-torus $\dim_2 H^*(F'; \mathbf{Z}/2) = 4 = \dim_2 H^*(X; \mathbf{Z}/2)$. Now $\mathbf{Z}/2 \approx T_2 \mid t^\perp$ acts on F' with fixed point set F. Because $\dim_2 H^*(F'; \mathbf{Z}/2) = \dim_2 H^*(F; \mathbf{Z}/2) = 4$, by Theorem VII-1-6 in [**3**, p. 374], $\mathbf{Z}/2$ acts trivially on $H^*(F'; \mathbf{Z}/2)$ and F' is totally nonhomologous to 0 in $F'_{\mathbf{Z}/2} \to B_{\mathbf{Z}/2}$. Consequently, T_2 acts trivially on $H^*(F'; \mathbf{Z}/2)$.

Because $\tilde{x}$ is the unique lift of x with the property stated earlier, we see that the coefficients in $j^*(\tilde{x})$ are polynomials in $t_1, \ldots, t_m$ invariant under even permutations. Let $\theta = \prod_{i<j}(t_i + t_j)$ and $t = t_1 + t_2$.

With these preliminary remarks we may start the proof of the main theorem. We use the theorem of J. C. Su **[10]** to obtain the possibilities for $H^*(F; \mathbf{Z}/2)$ and $H^*(F(t^\perp, X); \mathbf{Z}/2)$. Then we apply the topological splitting principle and Theorem IV.1 in **[9]** to obtain contradictions in each case.

Case 1. $F \sim_{\mathbf{Z}/2} 4$ points q_1, q_2, q_3, q_4.

$\dim F' = 1 + \dim F = 1$ and so it cannot consist of 4 points. In fact, $F' \approx_{\mathbf{Z}/2} S^p \cup S^{p'}$, $p \leqslant p'$, is the only possibility.

Let the connected components of F' be F_1' and F_2'. We may assume that $q_1 \in F_1'$. T_2 acts on F' and must send F_1' to F_1' and F_2' to F_2'. F_1' is a $\mathbf{Z}/2$-cohomology sphere, hence $F \cap F_1'$ has to be a $\mathbf{Z}/2$-cohomology sphere by Smith theory. Let $F \cap F_1' = \{q_1, q_2\}$. (Actually, F_1' and F_2' are diffeomorphic to circles.) Let $H^*(F_1': \mathbf{Z}/2) = \mathbf{Z}/2[z]/(z^2)$ and $\tilde{z}$ be a lift of z to $H^*_{T_2}(F_1'; \mathbf{Z}/2)$ so that under $H^*_{T_2}(F_1'; \mathbf{Z}/2) \to H^*_{T_2}(q_1 \cup q_2; \mathbf{Z}/2) \to H^*_{T_2}(q_1; \mathbf{Z}/2)$, $\tilde{z}$ goes to 0. We have a commutative diagram with self-explanatory notation.

$$\begin{array}{ccc} \tilde{x} \in H^*_{T_2}(X; \mathbf{Z}/2) & \to & \bigoplus_{1}^{4} H^*_{T_2}(q_i; \mathbf{Z}/2) \ni (0, \alpha_1, \alpha_2, \alpha_3) \\ & j_2^* \searrow \qquad \nearrow j_1^* & \\ & H^*_{T_2}(F_1'; \mathbf{Z}/2) \oplus H^*_{T_2}(F_2'; \mathbf{Z}/2) & \end{array}$$

Now $j_2^*(\tilde{x}) = (a_0 + a_1\tilde{z}, *)$ and $j_1^*(\tilde{z}, 0) = (0, t, 0, 0)$. Hence $a_0 = 0$, $a_1 t = \alpha_1$. It follows that θ divides α_1.

Next we apply the topological splitting principle. Let $j^*(b_0 + b_1\tilde{x} + b_2\tilde{y} + b_3\tilde{x}\tilde{y}) = \theta(0,1,0,0)$. Then $b_0 = 0$, $b_1\alpha_1 + b_2 \operatorname{Sq}^2 \alpha_1 + b_3\alpha_1 \operatorname{Sq}^2 \alpha_1 = \theta$. The second and third terms must be 0 since θ divides α_1. Hence $b_1\alpha_1 = \theta$, whence $b_1 = 1$. $\dim \tilde{x} = \dim \theta$ and $b_2 = 0 = b_3$. But then $j^*(\tilde{x}) = (0, \theta, 0, 0)$, a contradiction to $j^*(\tilde{x}) = (0, \alpha_1, \alpha_2, \alpha_3)$, where $\alpha_1, \alpha_2, \alpha_3$ are nonzero according to Theorem IV-1 in [9].

Case 2. $F \sim_{\mathbf{Z}/2} S^p \cup S^p, p \geq 1$.

Case 3. $F \sim_{\mathbf{Z}/2} \mathbf{Z}/2[u]/(u^4)$.

Case 4. $F \sim_{\mathbf{Z}/2} \mathbf{Z}/2[u, v]/(u^2 = v^2 \neq 0, uv = 0)$, $\dim u = 1, 2, 4, 8$.

Case 5. $F \sim_{\mathbf{Z}/2} \Lambda_{\mathbf{Z}/2}(f_1, f_2)$.

These cases are ruled out in the same way as in Case 1. We shall omit the detailed computations for Cases 2, 3, and 4.

Case 5. $F \sim_{\mathbf{Z}/2} \Lambda(f_1, f_2)$.

1. $F \sim_{\mathbf{Z}/2} \mathbf{Z}/2[u]/(u^4)$.

Let $j_1^*(\tilde{u}) = af_1 + bf_2 + cf_1f_2$. Then $j_1^*(\tilde{u}^2) = 0$, contradicting $\tilde{u}^2 \neq 0$ and the injectivity of j_1^*.

2. $F' \sim_{\mathbf{Z}/2} \mathbf{Z}/2[u, v]/(u^2 = v^2 \neq 0, uv = 0)$, $\dim u = 1, 2, 4, 8$. The same proof as in (1) applies to this case.

3. $F' \sim_{\mathbf{Z}/2} \Lambda(f_1', f_2')$.

Let $j^*(\tilde{x}) = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_1 f_2$ and $j_2^*(\tilde{x}) = a_0 + a_1\tilde{f}_1' + a_2\tilde{f}_2' + a_3\tilde{f}_1'\tilde{f}_2'$. Also, $j_1^*(\tilde{f}_1') = e_1 t^{c_1} f_1 + e_2 t^{c_2} f_2$, $j_1^*(\tilde{f}_2') = h_1 t^{d_1} f_1 + h_2 t^{d_2} f_2 + h_3 t^{d_3} f_1 f_2$, $j_1^*(\tilde{f}_1'\tilde{f}_2') = (e_1 h_2 t^{c_1 + d_2} + e_2 h_1 t^{d_1 + c_2}) f_1 f_2$. Therefore, $e_1 h_2 + e_2 h_1 = 1$. A simple calculation shows that if $\operatorname{Sq}^2 f_1 = f_2$, then $\alpha_1^2 + \alpha_1 \operatorname{Sq}^2 \alpha_2 + \alpha_2 \operatorname{Sq}^2 \alpha_1 = \theta$; if $\operatorname{Sq}^2 f_1 = 0$, then $\alpha_1 \operatorname{Sq}^2 \alpha_2 + \alpha_2 \operatorname{Sq}^2 \alpha_1 = \theta$ or $\alpha_1 \operatorname{Sq}^2 \alpha_1 + \alpha_1 \operatorname{Sq}^2 \alpha_2 + \alpha_2 \operatorname{Sq}^2 \alpha_1 = \theta$ depending on whether $\operatorname{Sq}^1 f_1 = 0$ or f_2. Also, $a_0 = 0$, $\alpha_1 = e_1 t^{c_1} a_1 + h_1 t^{d_1} a_2$, $\alpha_2 = e_2 t^{c_2} a_1 + h_2 t^{d_2} a_2$, $\alpha_3 = h_3 t^{d_3} a_2 + a_3 t$.

(a) $e_1 h_2 = 1$, $e_2 h_1 = 0$, $c_1 + d_2 = 1$.

(i) $c_1 = 0$, $d_2 = 1$.

If $e_2 = 0$, then θ divides α_2. The only possibility is $\alpha_1 \operatorname{Sq}^1 \alpha_1 = \theta$. It follows that $\operatorname{Sq}^1 f_1 = f_2$.

Also, since $j_1^*(\tilde{f}_1') = f_1$, then $j_1^*(\mathrm{Sq}^1 \tilde{f}_1') = f_2$. But $\mathrm{Sq}^1 \tilde{f}_1' = et\tilde{f}_1'$ and $j^*(\mathrm{Sq}^1 \tilde{f}_1') = etf_1 \neq f_2$. Therefore, $e_2 = 1$ holds. This forces $h_1 = 0$, $\dim f_1 = \dim f_2$, $c_2 = 0$, $d_1 = 1$. The equation $\alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta$ must therefore hold. $\alpha_1 = a_1$ and $\alpha_2 = a_1 + ta_2$, so $\alpha_1 + \alpha_2 = ta_2$. Again since $\alpha_1 \neq \alpha_2$, $\dim(\alpha_1 + \alpha_2) \geqslant \dim \theta$ and we get a contradiction since one of the terms $\alpha_1 \mathrm{Sq}^2 \alpha_2$ or $\alpha_2 \mathrm{Sq}^2 \alpha_1$ must be nonzero.

(ii) $c_1 = 1$, $d_2 = 0$.

In this case $d_1 \geqslant c_1 = 1$ forces θ to divide α_1. None of the equations can be satisfied.

(b) $e_1 h_2 = 0$, $e_2 h_1 = 1$, $d_1 + c_2 = 1$.

(i) $d_1 = 1$, $c_2 = 0$.

Either $\dim f_1' = \dim f_2'$ or $\dim f_1 = \dim f_2$. In the first case, $e_1 = 1$ and so θ divides α_1; hence none of the equations can be satisfied. In the second case, $c_1 = 0$, $d_2 = 1$ and $\alpha_1 \mathrm{Sq}^2 \alpha_2 + \alpha_2 \mathrm{Sq}^2 \alpha_1 = \theta$ must hold. $\alpha_1 = ta_2 + e_1 a_1$ and $\alpha_2 = a_1 + h_2 ta_2$. If $e_1 = 0$, θ divides α_1 and we obtain a contradiction as before. Otherwise $e_1 = 1$, $h_2 = 0$. Then $\alpha_1 + \alpha_2 = ta_2$ and we obtain θ to divide $\alpha_1 + \alpha_2$, leading to a contradiction as in (a)(i).

(ii) $d_1 = 0$, $c_2 = 1$.

This implies that $\dim F = \dim F'$, a contradiction.

The proof of the main theorem for $SO(m)$ is therefore complete. □

5. The case of $\mathrm{Spin}(m)$ **and concluding remarks.** If $\mathrm{Spin}(m)$ acts smoothly on $W_{n,2}$, n odd, with $(H^0) = (T)$ then the reduced geometric weight system equals the root system of $\mathrm{Spin}(m)$. $\{\pm 1\} \subseteq \mathrm{center}(\mathrm{Sp}(m))$ can then be shown to act trivially on $W_{n,2}$ using the denseness of principal orbits. In other words, the $\mathrm{Spin}(m)$ action factors through $SO(m)$ to give an action with $(H^0) = (T)$. Hence if T_2 is a maximal 2-torus of $SO(m)$ and $F(T_2, X) \neq \varnothing$ for the induced $SO(m)$ action, we are reduced to the analysis in §4.

Next we remark about the proof of the main corollary. There are two cases:

1. $SU(m)$ *case*. When $m \geqslant 5$ and $SU(m)$ acts in such a way that $F(T, X) \neq \varnothing$ and $(H^0) = (SU(2) \times \cdots \times SU(2))$ ($[m/2]$ factors), it follows that the reduced geometric weight system is $\{\pm(\theta_i + \theta_j)\}$, $i < j$.

It remains to observe that when restricted to the standard maximal 2-torus in $SU(m)$, $\{\pm(\theta_i + \theta_j)\}$ and $\{\pm(\theta_i - \theta_j)\}$ give the same 2-weights. Since the arguments in § 2 were in mod 2, exactly the same proof rules out the possibility of such an action.

2. $\mathrm{Sp}(m)$ *case*. When $m \geqslant 5$ and $\mathrm{Sp}(m)$ acts in such a way that $F(T, X) \neq \varnothing$ and $(H^0) = (\mathrm{Sp}(1) \times \cdots \times \mathrm{Sp}(1))$ (m factors), it follows that the reduced geometric weight system is $\{\pm(\theta_i + \theta_j), i < j\} = \Omega'([\Lambda^2 \nu_m]_{\mathbf{R}})$. We use the notation established in §3 in the following.

First observe that the action factors through center $(\mathrm{Sp}(m)) \approx \mathbf{Z}/2$. Let Q_2 denote the 2-torus (maximal in $\bar{T}$) generated by $\bar{\tau}_1, \ldots, \bar{\tau}_{m-1}, \bar{\delta}$. By Lemma 3-5, $F(T, X) = F(Q_2, X)$ is a $\mathbf{Z}/2$ cohomology product of two odd spheres. Let $H^*(F(T, X); \mathbf{Z}/2) \approx \Lambda(f_1, f_2)$, $\dim f_1 \leqslant \dim f_2$. Note that $\dim F(T, X) = 4n - 4 - 2m(m-1) \equiv 0$ (4). Hence $\dim f_2 - \dim f_1 \geqslant 2$, so that $N(Q_2)/Q_2$ acts trivially on $H^*(F(T, X); \mathbf{Z}/2)$. Let $t_1, \ldots, t_m$ be variables dual to $\bar{\tau}_1, \ldots, \bar{\tau}_{m-1}, \bar{\delta}$. We set up the usual commutative diagram

$$\begin{array}{ccc} H^*_{Q_2}(X;\mathbf{Z}/2) & \overset{j^*}{\rightarrowtail} & H^*_{Q_2}(F;\mathbf{Z}/2) \\ j_2^*\searrow & & \nearrow j_1^* \\ & H^*_{Q_2}(F';\mathbf{Z}/2) & \end{array}$$

where $F' = F(t^\perp, X)$ and j^*, j_1^* are monomorphisms. Let $j^*(\tilde{x}) = af_1 + bf_2 + cf_1f_2$. Then a, b, c are invariant under the action of $N(Q_2)/Q_2$.

By Lemma 3-6, the system of 2-weights at $F(Q_2, X)$ is given by $2\{t_i,\ 1 \leq i \leq m-1;\ t_i + t_j,\ 1 \leq i < j \leq m-1;\ t_i + t_m,\ 1 \leq i \leq m-1;\ t_i + t_j + t_m,\ 1 \leq i < j \leq m-1\}$. Let

$$\theta = \left(\prod_{i=1}^{m-1} t_i(t_i + t_m)\right)\left(\prod_{1 \leq i < j \leq m-1} (t_i + t_j)(t_i + t_j + t_m)\right).$$

By Lemma 3-7, the 2-weights appearing in θ form a single orbit under the action of $N(Q_2)/Q_2$.

For $t = t_1$, observe that $t_1^\perp \subset \overline{(\theta_1 - \theta_m)}^\perp$ so that $F(t_1^\perp, X) = F((\theta_1 - \theta_m)^\perp, X)$ is a $\mathbf{Z}/2$ cohomology product of two odd spheres.

Using analysis as that in the proof of the main theorem for $\mathrm{Sp}(m)$ we conclude that such a weight system is impossible.

Lastly, we make a few remarks concerning the status of the main theorem for $m \leq 4$. For $\mathrm{SU}(2) = \mathrm{Sp}(1)$, we have

PROPOSITION. $\mathrm{SU}(2) = \mathrm{Sp}(1)$ *and* $\mathrm{SO}(3)$ *cannot act smoothly on* $W_{n,2}$, *n odd, with* $(H^0) = (T)$.

PROOF. Note that we may take $T = \mathrm{SO}(2)$. Let $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$. First note that if $G_x \neq G$, then G_x^0 is a rank 1 compact connected Lie subgroup of G and so must be a circle subgroup.

Now let $x \in F(T, X)$. If $x \in F(G, X)$, then $\Omega'_T(X) = \Omega'_T(S_x)$ where S_x is a slice at x. Since G acts orthogonally on S_x with connected principal isotropy type (T), $\Omega'_T(S_x) = \Delta(G)$. If $x \in F(T, X) - F(G, X)$, then $T \subseteq G_x$ acts trivially on S_x (since the connected principal isotropy type is a circle). Hence $\Omega'_T(X) = \Delta(G) - \Delta(T) = \Delta(G)$.

Note that the action factors through center (G) if $G = \mathrm{SU}(2) \approx \mathrm{Sp}(1)$. In other words, it induces an $\mathrm{SO}(3)$-action with $(H^0) = (\mathrm{SO}(2))$. Hence we need only show that no such $\mathrm{SO}(3)$ action exists.

Restrict the $\mathrm{SO}(3)$ action to $\mathrm{SO}(2)$. $F = F(\mathrm{SO}(2), X)$ is an integral cohomology product of two odd spheres. Suppose $H^*(F;\mathbf{Z}/2) \approx \Lambda(f_1, f_2)$. Let $\pm\theta$ be the root of $\mathrm{SO}(3)$. Then the arguments in this section show that if $j^*\colon H^*_{\mathrm{SO}(2)}(X;\mathbf{Z}/2) \to H^*_{\mathrm{SO}(2)}(F;\mathbf{Z}/2)$ sends $\tilde{x}$ to $af_1 + bf_2$, $a, b \in H^*(B\,\mathrm{SO}(2);\mathbf{Z}/2) = \mathbf{Z}/2[\theta]$, then we get equations $a^2 + a\,\mathrm{Sq}^2 b + b\,\mathrm{Sq}^2 a = \theta$ if $\mathrm{Sq}^2 f_1 = f_2$ and $a\,\mathrm{Sq}^2 b + b\,\mathrm{Sq}^2 a = \theta$ if $\mathrm{Sq}^2 f_1 = 0$. But none of these can be satisfied. $\square$

The remaining cases of $\mathrm{SU}(3)$, $\mathrm{SU}(4)$, $\mathrm{Sp}(2)$, $\mathrm{Sp}(3)$, $\mathrm{Sp}(4)$ probably require separate arguments as in the above proposition. Note that when $F(T_2, X) \neq \varnothing$, then the cases $\mathrm{SU}(4) = \mathrm{Spin}(6)$ and $\mathrm{Sp}(2) = \mathrm{Spin}(5)$ can be ruled out by results in §4.

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